



Extremal values of multiple gamma and sine functions

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Abstract

We study the extremal values of multiple gamma and sine functions in the fundamental intervals. We show the number and locations of the extremal points, and prove that all the local maximum and minimum values are greater and less than one, respectively.

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1. Introduction

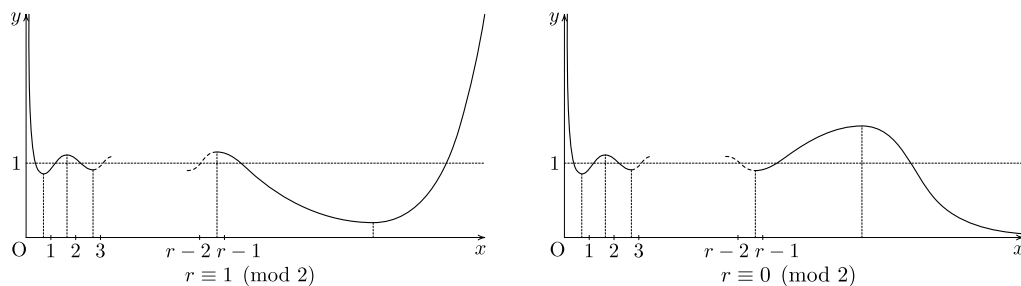
In 1904 Barnes [1] introduced the multiple gamma function

$$\Gamma_r(x) = \exp\left(\frac{\partial}{\partial s} \zeta_r(s, x) \Big|_{s=0}\right) \quad (x > 0)$$

where $\zeta_r(s, x)$ is the multiple Hurwitz zeta function defined as

$$\zeta_r(s, x) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{1}{(n_1 + \dots + n_r + x)^s} \quad (\operatorname{Re}(s) > r).$$

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Fig. 1. The graph of $\Gamma_r(x)$.

He studied the behavior of $\Gamma_r(x)$ for large positive values of x and obtained the asymptotic expansion of $\log \Gamma_r(x)$ [1, p. 424]:

$$\begin{aligned} \log \Gamma_r(x) = & \sum_{n=0}^r \frac{(-1)^{r-1}}{n!(r-n)!} B_n^{(r)} (\log x - H_{r-n}) x^{r-n} \\ & + \sum_{n=r+1}^{r+N} \frac{(-1)^n (n-r-1)!}{n!} B_n^{(r)} x^{r-n} + O(x^{-N-1}) \end{aligned}$$

where $B_n^{(r)}$ is the Nörlund polynomial and H_n is the n -th harmonic number. However, little is known about the behavior of $\Gamma_r(x)$ for small positive values of x . Therefore we study the extremal values of $\Gamma_r(x)$. In the simplest case $r = 1$, Lerch's formula shows that

$$\Gamma_1(x) = \frac{\Gamma(x)}{\sqrt{2\pi}}$$

where $\Gamma(x)$ is the usual gamma function. Hence, $\Gamma_1(x)$ has, in the interval $(0, +\infty)$, the unique extremal value $\Gamma_1(\alpha) = 0.35330\dots < 1$ with $\alpha = 1.46163\dots$. Our aim is to give its analogue for $\Gamma_r(x)$ ($r \geq 2$). We easily see that

$$\frac{d^{r+1}}{dx^{r+1}} \log \Gamma_r(x) = (-1)^{r+1} r! \zeta_r(r+1, x)$$

which implies that $\Gamma_r(x)$ has at most r extremal points in $(0, +\infty)$. In this paper, we obtain a more precise result as follows.

Theorem 1. *The multiple gamma function $\Gamma_r(x)$ has exactly r extremal points in the interval $(0, +\infty)$. To be precise, $\Gamma_r(x)$ has a local maximum (resp. minimum) point in $(j-1, j)$ for any even (resp. odd) integer j with $1 \leq j \leq r-1$, and an extremal point in $(r-1, +\infty)$. Moreover, all the local maximum (resp. minimum) values are greater (resp. less) than one. The graph of $\Gamma_r(x)$ is as in Fig. 1.*

Remark. Barnes originally called $\Gamma_r^B(x) = \rho_r \Gamma_r(x)$ the multiple gamma function, where ρ_r is the normalization factor called the r -ple Stirling modular form which yields $\Gamma_r^B(x) \sim 1/x$ as $x \rightarrow 0$. The function $\Gamma_r^B(x)$ also satisfies the same properties for the number and locations of the

extremal points as in Theorem 1. Moreover, all the local maximum values are greater than one, since $\rho_r > 1$ (see Corollary 4.2). However, its local minimum value is not always less than one. For instance, $\Gamma_2^B(x)$ has the local minimum value $\Gamma_2^B(\beta) = 2.3470\dots$ with $\beta = 0.75796\dots$

Our second aim is to investigate the extremal values of the multiple sine function

$$S_r(x) = \Gamma_r(x)^{-1} \Gamma_r(r-x)^{(-1)^r} \quad (0 < x < r)$$

which originated with Kurokawa [5]. Regarding the multiple sine function $S_r(x, \omega)$ with $r \leq 5$ and general quasi-period parameters ω , we have already studied its behavior in [3,4,6,10,12]. In this paper, we consider the case that r is unconditional and $\omega = 1$.

By the logarithmic derivative of $S_r(x)$ (Proposition 2.2), we have already known that $S_r(x)$ has exactly r extremal points in the interval $(0, r)$, and

$$S_r\left(k + \frac{1}{2}\right) = 2^{(-1)^{r-1} \binom{k-1/2}{r-1}} \times \exp\left(\sum_{m=1}^{\lfloor (r-1)/2 \rfloor} \frac{(-1)^{r-m-1} (1-2^{-2m}) B_{r-2m-1}^{(r)}(k+1/2)}{(2\pi)^{2m} (r-2m-1)!} \zeta(2m+1)\right)$$

for $k = 0, 1, \dots, r-1$ (see [12, Theorem 3]) is a local maximum (resp. minimum) value, if k is an even (resp. odd) integer. Here $B_m^{(r)}(x)$ is the generalized Bernoulli polynomial given by

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{m=0}^{\infty} \frac{B_m^{(r)}(x)}{m!} t^m.$$

Hence, we concentrate on the sizes of the extremal values. Our results are as follows.

Theorem 2. *In the interval $(0, r)$, all the local maximum (resp. minimum) values of $S_r(x)$ are greater (resp. less) than one.*

Theorem 3. *Let k and l be integers in $[0, r-1]$. If $|k - r/2| > |l - r/2|$, then*

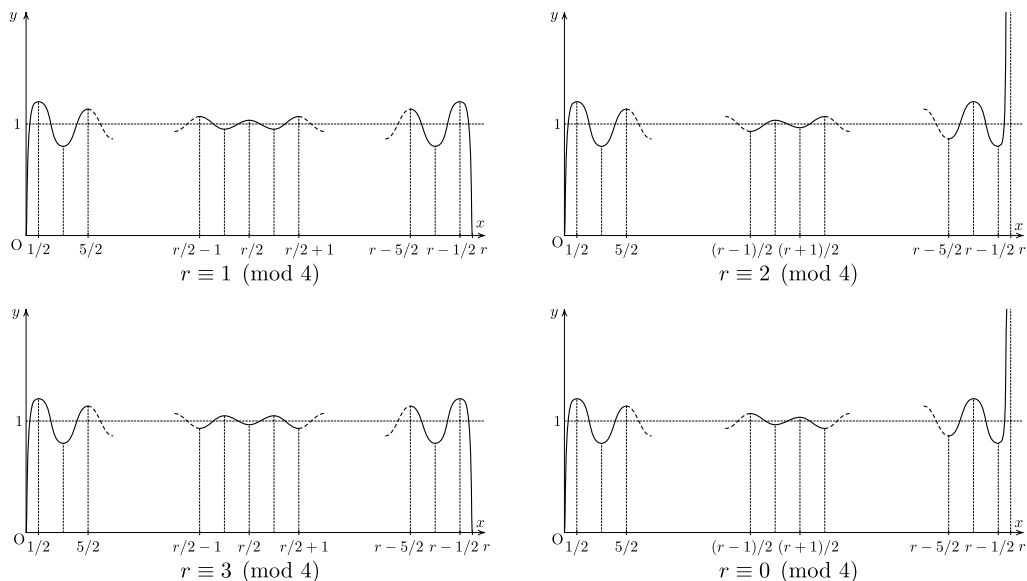
$$\left|\log S_r\left(k + \frac{1}{2}\right)\right| > \left|\log S_r\left(l + \frac{1}{2}\right)\right|.$$

Corollary. *If r is an odd integer ≥ 1 , then $S_r(1/2)(= S_r(r-1/2))$ is the maximum value of $S_r(x)$ in the interval $(0, r)$.*

The behavior of $S_r(x)$ is shown in Fig. 2.

2. Logarithmic derivatives of multiple gamma and sine functions

We express the logarithmic derivatives of multiple gamma and sine functions by well-known functions, which is essential to prove the theorems.

Fig. 2. The graph of $S_r(x)$.

Proposition 2.1. Let r be a positive integer and set $\psi(x) = (\Gamma'/\Gamma)(x)$. Then we have

$$\frac{\Gamma'_r}{\Gamma_r}(x) = (-1)^{r-1} \binom{x-1}{r-1} \psi(x) + F_{r-1}(x), \quad (2.1)$$

where $F_0(x) = 0$ and $F_r(x)$ for $r \geq 1$ is a polynomial of degree r given by

$$F_r(x) = \frac{(-1)^{r+1}}{r!} \sum_{j=1}^r \frac{B_j^{(j)}(x)}{j} (x-r)_{r-j}. \quad (2.2)$$

Here $(x)_n$ is the Pochhammer symbol.

Results of this kind have been proved by Kanemitsu, Kumagai and Yoshimoto [2, Theorem 5] and Nishizawa [8, Proposition 9(ii)]. Our result gives the simplest expression for $F_r(x)$.

Proof of Proposition 2.1. When $r = 1$, the result is trivial. We treat with the case $r \geq 2$. Note that

$$\zeta_r(s, x) = \frac{1}{r-1} \zeta_{r-1}(s-1, x) - \frac{x-r+1}{r-1} \zeta_{r-1}(s, x).$$

Hence

$$\log \Gamma_r(x) = \frac{1}{r-1} \frac{\partial}{\partial s} \zeta_{r-1}(s-1, x) \Big|_{s=0} - \frac{x-r+1}{r-1} \log \Gamma_{r-1}(x).$$

Since

$$\begin{aligned} \left. \frac{\partial^2}{\partial x \partial s} \zeta_{r-1}(s-1, x) \right|_{s=0} &= \frac{\partial}{\partial s} \left\{ -(s-1) \zeta_{r-1}(s, x) \right\} \Big|_{s=0} \\ &= -\zeta_{r-1}(0, x) + \log \Gamma_{r-1}(x) \\ &= \frac{(-1)^r B_{r-1}^{(r-1)}(x)}{(r-1)!} + \log \Gamma_{r-1}(x), \end{aligned}$$

we obtain

$$\frac{\Gamma'_r}{\Gamma_r}(x) = -\frac{x-r+1}{r-1} \frac{\Gamma'_{r-1}}{\Gamma_{r-1}}(x) + \frac{(-1)^r B_{r-1}^{(r-1)}(x)}{(r-1) \cdot (r-1)!} \quad (2.3)$$

for $r \geq 2$. Using this formula repeatedly, we have (2.1). We next confirm $\deg F_r = r$ for $r \geq 1$. Note that the term of highest degree in $B_j^{(j)}(x)$ is x^j . Hence, Eq. (2.2) shows that

$$F_r(x) = \frac{(-1)^{r+1} H_r}{r!} x^r + O(|x|^{r-1}) \quad (|x| < 1), \quad (2.4)$$

where H_r is the r -th harmonic number. This completes the proof of Proposition 2.1. \square

Proposition 2.2. (See Kurokawa and Koyama [7, Theorem 2.15].) *Let r be a positive integer. Then*

$$\frac{S'_r}{S_r}(x) = (-1)^{r-1} \binom{x-1}{r-1} \pi \cot(\pi x). \quad (2.5)$$

Kurokawa and Koyama proved this formula via the differential equation of the primitive multiple sine function $\mathcal{S}_r(x)$:

$$\frac{\mathcal{S}'_r}{\mathcal{S}_r}(x) = \pi x^{r-1} \cot(\pi x)$$

(see [7, Theorem 2.5]). We here give a direct proof of (2.5).

Proof of Proposition 2.2. When $r = 1$, the formula follows from $S_1(x) = 2 \sin(\pi x)$. If $r \geq 2$, then we can see that

$$\zeta_r(s, x+1) = \frac{1}{r-1} \zeta_{r-1}(s-1, x) - \frac{x}{r-1} \zeta_{r-1}(s, x)$$

and

$$\frac{\Gamma'_r}{\Gamma_r}(r-x) = \frac{x-r+1}{r-1} \frac{\Gamma'_{r-1}}{\Gamma_{r-1}}(r-1-x) + \frac{(-1)^r B_{r-1}^{(r-1)}(r-1-x)}{(r-1) \cdot (r-1)!}. \quad (2.6)$$

Since $B_m^{(r)}(r-x) = (-1)^m B_m^{(r)}(x)$ for $r \geq 1$ and $m \geq 0$, the formulas (2.3) and (2.6) show

$$\frac{S'_r}{S_r}(x) = -\frac{x-r+1}{r-1} \frac{S'_{r-1}}{S_{r-1}}(x).$$

Thus, the formula (2.5) follows. \square

3. Properties of the polynomial $F_r(x)$

We first investigate the sign of special values of the polynomial $F_r(x)$. The next result will show the number of extremal points of $F_r(x)$.

Proposition 3.1. *Let r be a positive integer.*

- (i) $\text{sign } F_r(k) = (-1)^{k+1}$ for $k = 0, \dots, r$.
- (ii) If $r \geq 2$, then $\text{sign } F_r(k - 1/2) = (-1)^k$ for $k = 1, \dots, r$.

Remark. $F_1(x) = x - 1/2$.

Lemma 3.2. *Let r be a positive integer.*

- (i) $B_r^{(r)}(x) > 0$ in $[r, \infty)$ and $\text{sign } B_r^{(r)}(k) = (-1)^{r+k}$ for $k = 0, \dots, r$.
- (ii) For $k = 1, \dots, r$,

$$\text{sign } B_r^{(r)}\left(k - \frac{1}{2}\right) = \begin{cases} (-1)^{r+k+1} & \text{if } k < (r+1)/2, \\ 0 & \text{if } k = (r+1)/2, \\ (-1)^{r+k} & \text{if } k > (r+1)/2. \end{cases}$$

To prove this lemma, we need the following formulas:

$$B_m^{(r)}(x+1) = B_m^{(r)}(x) + m B_{m-1}^{(r-1)}(x) \quad (r \geq 2, m \geq 1),$$

$$\frac{d}{dx} B_m^{(r)}(x) = m B_{m-1}^{(r)}(x) \quad (r \geq 1, m \geq 1),$$

$$B_{r-1}^{(r)}(x) = (x-r+1)_{r-1} \quad (r \geq 1).$$

These properties of $B_m^{(r)}(x)$ are found in the book of Nörlund [9].

Proof of Lemma 3.2. (i) By the above formulas, we have

$$\begin{aligned} B_r^{(r)}(x) &= \frac{1}{r+1} (B_{r+1}^{(r+1)}(x+1) - B_{r+1}^{(r+1)}(x)) \\ &= \int_x^{x+1} B_r^{(r+1)}(t) dt \\ &= \int_x^{x+1} (t-r)_r dt. \end{aligned} \tag{3.1}$$

Hence, the first assertion is true and

$$B_r^{(r)}(k) = (-1)^{r+k} \int_k^{k+1} |(t-r)_r| dt \quad (k=0, \dots, r)$$

which shows the second result.

(ii) We first assume that $k \leq (r+1)/2$. By (3.1) we obtain

$$\begin{aligned} B_r^{(r)}\left(k - \frac{1}{2}\right) &= \int_{k-1/2}^{k+1/2} (t-r)_r dt \\ &= \int_0^{1/2} \{(t+k-r)_r + (-t+k-r)_r\} dt \\ &= (-1)^{r+k+1} \int_0^{1/2} (t)_k (1-t)_{k-1} \{(k+t)_{r-2k+1} - (k-t)_{r-2k+1}\} dt. \end{aligned}$$

Thus, $\text{sign } B_r^{(r)}(k-1/2) = (-1)^{r+k+1}$ if $k < (r+1)/2$, and $B_r^{(r)}(k-1/2) = 0$ if $k = (r+1)/2$.

Next we suppose that $k > (r+1)/2$. Since $B_r^{(r)}(k-1/2) = (-1)^r B_r^{(r)}(r-k+1-1/2)$, we have $\text{sign } B_r^{(r)}(k-1/2) = (-1)^r \cdot (-1)^{r+(r-k+1)+1} = (-1)^{r+k}$. \square

Proof of Proposition 3.1. (i) By Lemma 3.2, we have

$$\begin{aligned} F_r(k) &= \frac{(-1)^{r+1}}{r!} \sum_{j=\max\{k,1\}}^r \frac{B_j^{(j)}(k)}{j} (k-r)_{r-j} \\ &= \frac{(-1)^{k+1}}{r!} \sum_{j=\max\{k,1\}}^r \left| \frac{B_j^{(j)}(k)}{j} (k-r)_{r-j} \right|. \end{aligned}$$

(ii) Since

$$F_r\left(k - \frac{1}{2}\right) = \frac{(-1)^k}{r!} \left(\sum_{j=1}^{k-1} - \sum_{j=k}^{\min\{2k-2,r\}} + \sum_{j=2k}^r \right) \left| \frac{B_j^{(j)}(k-1/2)}{j} \left(k - \frac{1}{2} - r\right)_{r-j} \right|,$$

the assertion is true when $k=1$, and for $k \geq 2$ it is enough to show

$$\begin{aligned} &\text{sign} \left\{ j^{-1} B_j^{(j)} \left(k - \frac{1}{2}\right) \left(k - \frac{1}{2} - r\right)_{r-j} \right. \\ &\quad \left. + (j+k-1)^{-1} B_{j+k-1}^{(j+k-1)} \left(k - \frac{1}{2}\right) \left(k - \frac{1}{2} - r\right)_{r-j-k+1} \right\} = (-1)^{r+k+1} \end{aligned} \quad (3.2)$$

for $1 \leq j \leq \min\{k-1, r-k+1\}$. Since

$$B_j^{(j)}\left(k - \frac{1}{2}\right) = \int_0^{1/2} \{(t+k-j)_j + (-t+k-j)_j\} dt,$$

$$B_{j+k-1}^{(j+k-1)}\left(k - \frac{1}{2}\right) = (-1)^{j+1} \int_0^{1/2} \{(t)_k(1-t)_{j-1} - (t)_j(1-t)_{k-1}\} dt,$$

$(k-1/2-r)_{r-j} = (-1)^{r+k+1}(j+1/2)_{r-j-k+1}(1/2)_j(1/2)_{k-j-1}$, and $(k-1/2-r)_{r-j-k+1} = (-1)^{r+j+k+1}(j+1/2)_{r-j-k+1}$, the content of the sign on the left side of (3.2) is

$$= (-1)^{r+k+1} \left(j + \frac{1}{2}\right)_{r-j-k+1} \int_0^{1/2} G(k, j; t) dt$$

where

$$G(k, j; t) := j^{-1}(1/2)_{k-j-1}(1/2)_j \{(t+k-j)_j + (-t+k-j)_j\} \\ - (j+k-1)^{-1} \{(t)_k(1-t)_{j-1} - (t)_j(1-t)_{k-1}\}.$$

We now show that $G(k, j; t) > 0$ when $k \geq 2$, $1 \leq j \leq k-1$, and $0 < t < 1/2$. We can see that $(t)_k(1-t)_{j-1}$ is monotonically increasing in $t \in (0, 1/2)$, since

$$\frac{d}{dt}(t)_k(1-t)_{j-1} = (t)_k(1-t)_{j-1}(\psi(1-t) - \psi(t) + \psi(t+k) - \psi(j-t)) > 0.$$

Hence

$$G(k, j; t) > 2j^{-1}(1/2)_{k-j-1}(1/2)_j(k-j-1/2)_j - (j+k-1)^{-1}(1/2)_k(1/2)_{j-1} \\ = 2j^{-1}(1/2)_{k-j-1}(1/2)_j(k-j-1/2)_j \\ - (j+k-1)^{-1}(k-j-1/2)_{j+1}(1/2)_{k-j-1}(1/2)_{j-1} \\ = \frac{2(j-1)k + 4j^2 - 5j + 2}{2j(k+j-1)} \cdot \left(\frac{1}{2}\right)_{k-j-1} \left(\frac{1}{2}\right)_{j-1} \left(k-j-\frac{1}{2}\right)_j \\ > 0.$$

Thus, we obtain $\text{sign } F_r(k-1/2) = (-1)^k$. \square

The next result is useful to estimate the extremal values of $\Gamma_r(x)$.

Proposition 3.3. *Let r be a positive integer ≥ 2 .*

- (i) *The polynomial $F_r(x)$ has zeros at $x = \alpha_{r,j} \in (j-1/2, j)$ for $j = 1, \dots, r$.*
- (ii) *$\alpha_{r,j} + 1 < \alpha_{r,j+1}$ for $j = 1, \dots, r-1$.*

Remark. The polynomial $F_1(x) = x - 1/2$ has a zero at $x = \alpha_{1,1} = 1/2$.

Lemma 3.4. Let r be a positive integer and put $\tilde{F}_r(x) = (x - r)F_r(x + 1) - xF_r(x)$. The polynomial $\tilde{F}_r(x)$ has the following properties:

- (i) $\deg \tilde{F}_r = r - 1$.
- (ii) $\text{sign } \tilde{F}_r(k) = (-1)^k$ for $k = 1, \dots, r$.
- (iii) $\text{sign } \tilde{F}_r(k - 1/2) = (-1)^k$ for $k = 1, \dots, r$.

Proof. By simple calculation we have

$$\tilde{F}_r(x) = \frac{(-1)^r}{r!} \sum_{j=1}^r \tilde{B}_j^{(j)}(x)(x - r)_{r-j}$$

where

$$\begin{aligned} \tilde{B}_j^{(j)}(x) &= j^{-1} \{x B_j^{(j)}(x) - (x - j) B_j^{(j)}(x + 1)\} \\ &= \int_x^{x+1} (t - x)(t + 1 - j)_{j-1} dt. \end{aligned}$$

(ii) Since

$$\begin{aligned} \tilde{F}_r(k) &= \frac{(-1)^r}{r!} \sum_{j=k}^r \tilde{B}_j^{(j)}(k)(k - r)_{r-j}, \\ \tilde{B}_j^{(j)}(k) &= (-1)^{j+k} \int_0^1 (t)_k (-t)_{j-k} dt, \end{aligned}$$

and $(k - r)_{r-j} = (-1)^{r+j} (r - k)! / (j - k)!$, we obtain

$$\tilde{F}_r(k) = \frac{(-1)^k (r - k)!}{r!} \int_0^1 \left(\sum_{j=k}^r \frac{(-t)_{j-k}}{(j - k)!} \right) (t)_k dt.$$

We note that, for $0 < t < 1$,

$$\sum_{j=k}^r \frac{(-t)_{j-k}}{(j - k)!} = \sum_{j=0}^{r-k} \frac{(-t)_j}{j!} = - \sum_{j=r-k+1}^{\infty} \frac{(-t)_j}{j!} > 0.$$

Thus, we have $\text{sign } \tilde{F}_r(k) = (-1)^k$ for $k = 1, \dots, r$.

(i) Since $\deg F_r = r$, we have $\deg \tilde{F}_r \leq r - 1$. The result (ii) implies $\deg \tilde{F}_r \geq r - 1$. Hence $\deg \tilde{F}_r = r - 1$.

(iii) We first calculate that

$$\begin{aligned}\tilde{F}_r\left(k - \frac{1}{2}\right) &= \frac{(-1)^{k+1}}{r!} \sum_{j=1}^{k-1} \left(\frac{1}{2}\right)_{r-k+1} \left(\frac{1}{2}\right)_{k-j-1} \int_{-1/2}^{1/2} \left(t + \frac{1}{2}\right) (t+k+1-j)_{j-1} dt \\ &\quad + \frac{(-1)^k}{r!} \sum_{j=k}^r \left(\frac{3}{2} + j - k\right)_{r-j} \int_{-1/2}^{1/2} \left(t + \frac{1}{2}\right) (-t)_{j-k} (1+t)_{k-1} dt.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{(-1)^k r! \tilde{F}_r(k - 1/2)}{(1/2)_{r-k+1}} \\ &= - \sum_{j=1}^{k-1} \int_0^{1/2} \left(\frac{(1/2)_{j-1}}{(1+t)_j} + \frac{(1/2-t)(1-t)_{k-1}}{(1/2+t)(1+t)_{k-1}} \frac{(1/2)_{j-1}}{(1-t)_j} \right) \left(t + \frac{1}{2}\right) (t+1)_{k-1} dt \\ &\quad + \sum_{j=0}^{r-k} \int_0^{1/2} \left(\frac{(-t)_j}{(1/2)_{j+1}} + \frac{(1/2-t)(1-t)_{k-1}}{(1/2+t)(1+t)_{k-1}} \frac{(t)_j}{(1/2)_{j+1}} \right) \left(t + \frac{1}{2}\right) (t+1)_{k-1} dt \\ &= \int_0^{1/2} H(k, r-k+1; t) \left(t + \frac{1}{2}\right) (t+1)_{k-1} dt\end{aligned}$$

where for $m \geq 1$ and $n \geq 1$

$$\begin{aligned}H(m, n; t) &:= - \sum_{j=1}^{m-1} \frac{(1/2)_{j-1}}{(1+t)_j} - \frac{(1/2-t)(1-t)_{m-1}}{(1/2+t)(1+t)_{m-1}} \sum_{j=1}^{m-1} \frac{(1/2)_{j-1}}{(1-t)_j} \\ &\quad + \sum_{j=0}^{n-1} \frac{(-t)_j}{(1/2)_{j+1}} + \frac{(1/2-t)(1-t)_{m-1}}{(1/2+t)(1+t)_{m-1}} \sum_{j=0}^{n-1} \frac{(t)_j}{(1/2)_{j+1}}.\end{aligned}$$

Thus, it is sufficient for our purpose to show $H(m, n; t) > 0$ for $t \in (0, 1/2)$. Since

$$\sum_{j=1}^{\infty} \frac{(1/2)_{j-1}}{(1+u)_j} = \frac{1}{u+1/2}, \quad \sum_{j=0}^{\infty} \frac{(-u)_j}{(1/2)_{j+1}} = \frac{1}{u+1/2}$$

for $|u| < 1/2$, we have

$$\begin{aligned}H(m, n; t) &= \sum_{j=m}^{\infty} \frac{(1/2)_{j-1}}{(1+t)_j} + \frac{(1/2-t)(1-t)_{m-1}}{(1/2+t)(1+t)_{m-1}} \sum_{j=m}^{\infty} \frac{(1/2)_{j-1}}{(1-t)_j} \\ &\quad - \sum_{j=n}^{\infty} \frac{(-t)_j}{(1/2)_{j+1}} - \frac{(1/2-t)(1-t)_{m-1}}{(1/2+t)(1+t)_{m-1}} \sum_{j=n}^{\infty} \frac{(t)_j}{(1/2)_{j+1}}.\end{aligned}$$

We note that the difference

$$H(m, n+1; t) - H(m, n; t) = \frac{(-t)_n}{(1/2)_{n+1}} + \frac{(1/2-t)(1-t)_{m-1}}{(1/2+t)(1+t)_{m-1}} \cdot \frac{(t)_n}{(1/2)_{n+1}}$$

is monotonically decreasing in m . Hence, we show that for $m \geq n \geq 1$,

$$\begin{aligned} H(m, n+1; t) - H(m, n; t) &\leq \frac{(-t)_n}{(1/2)_{n+1}} + \frac{(1/2-t)(1-t)_{n-1}}{(1/2+t)(1+t)_{n-1}} \cdot \frac{(t)_n}{(1/2)_{n+1}} \\ &= -\frac{2t^2(1-t)_{n-1}}{(1/2+t)(1/2)_{n+1}} \\ &< 0, \end{aligned}$$

and that $H(m, n; t) > H(m, m+1; t)$ for $m \geq n \geq 1$. Therefore, we treat only the case $n \geq m+1$. Then

$$\begin{aligned} H(m, n; t) &> \frac{(1/2-t)(1-t)_{m-1}}{(1/2+t)(1+t)_{m-1}} \left(\sum_{j=m}^{\infty} \frac{(1/2)_{j-1}}{(1-t)_j} - \sum_{j=n}^{\infty} \frac{(t)_j}{(1/2)_{j+1}} \right) \\ &\geq \frac{(1/2-t)(1-t)_{m-1}}{(1/2+t)(1+t)_{m-1}} \sum_{j=m}^{\infty} \frac{(1/2)_{j-1}(1/2)_{j+2} - (t)_{j+1}(1-t)_j}{(1-t)_j(1/2)_{j+2}}. \end{aligned}$$

Since the function $(t)_{j+1}(1-t)_j$ is monotonically increasing in $t \in (0, 1/2)$, we obtain

$$(1/2)_{j-1}(1/2)_{j+2} - (t)_{j+1}(1-t)_j > 2(1/2)_{j-1}(1/2)_{j+1}$$

and $H(m, n; t) > 0$ for $n \geq m+1$. Thus, the result (iii) follows. \square

Proof of Proposition 3.3. Proposition 3.1 implies (i) and suggests that $x \in (j, \alpha_{r,j+1})$ for $j = 1, \dots, r-1$, if $x \in (j, j+1)$ and $\text{sign } F_r(x) = (-1)^{j+1}$. Note that $\text{sign } \tilde{F}_r(\alpha_{r,j}) = -\text{sign } F_r(\alpha_{r,j} + 1)$. Hence, we obtain $\text{sign } F_r(\alpha_{r,j} + 1) = (-1)^{j+1}$ by (i) and Lemma 3.4. Thus we have $\alpha_{r,j} + 1 < \alpha_{r,j+1}$ for $j = 1, \dots, r-1$. \square

4. Extremal values of multiple gamma function

In this section, we prove the following result which implies Theorem 1.

Proposition 4.1. Let r be a positive integer. Assume that $\alpha_{r,j}$ ($1 \leq j \leq r$) is same as in Proposition 3.3.

- (i) If x is a sufficiently small positive number, then $\Gamma'_r(x) < 0$.
- (ii) $\text{sign } \Gamma'_r(k) = (-1)^{k-1}$ for $k = 1, \dots, r-1$.
- (iii) If x is a sufficiently large positive number, then $\text{sign } \Gamma'_r(x) = (-1)^{r-1}$.
- (iv) $\log \Gamma_r(1) < 0$.
- (v) $\text{sign}(\log \Gamma_r(\alpha_{r,j})) = (-1)^j$ for $r \geq 2$ and $j = 2, \dots, r$.

Since $\Gamma_r(1) = \rho_{r-1}/\rho_r$ for $r \geq 1$ with $\rho_0 := 1$, we have the following.

Corollary 4.2. *The Stirling modular form ρ_r is monotonically increasing in r .*

Proof of Proposition 4.1. Proposition 2.1 shows that

$$\begin{aligned}\frac{\Gamma'_r}{\Gamma_r}(x) &\sim -\frac{1}{x} \quad \text{as } x \rightarrow 0+, \\ \frac{\Gamma'_r}{\Gamma_r}(k) &= F_{r-1}(k) \quad \text{for } k = 1, \dots, r-1, \\ \frac{\Gamma'_r}{\Gamma_r}(x) &\sim \frac{(-1)^{r-1}}{(r-1)!} x^{r-1} \log x \quad \text{as } x \rightarrow +\infty.\end{aligned}$$

Hence, (i) and (iii) hold, and (ii) follows from Proposition 3.1.

(iv) We have

$$\begin{aligned}\log \Gamma_r(x) &= \log \Gamma_{r+1}(x) - \log \Gamma_{r+1}(x+1) \\ &= - \int_x^{x+1} \frac{\Gamma'_{r+1}}{\Gamma_{r+1}}(t) dt \\ &= (-1)^{r-1} \int_x^{x+1} \binom{t-1}{r} \psi(t) dt - \int_x^{x+1} F_r(t) dt.\end{aligned}$$

Hence, $\log \Gamma_r(1) = -I - J$ where

$$I := (-1)^r \int_1^2 \binom{t-1}{r} \psi(t) dt, \quad J := \int_1^2 F_r(t) dt.$$

We remark that, by (2.4), the polynomial $F_r(t)$ can be factorized as

$$F_r(t) = \frac{(-1)^{r+1} H_r}{r!} (t - \alpha_{r,1}) \cdots (t - \alpha_{r,r}).$$

Note that $\psi(t)$ ($t > 0$) is monotonically increasing and has a zero at $t = \alpha = 1.46163\dots$. Suppose that $r \geq 4$. Then we have

$$\begin{aligned}r!I &= - \left(\int_1^\alpha + \int_\alpha^2 \right) (t-1)(2-t) \cdots (r-t) \psi(t) dt \\ &\geq - \int_1^2 (t-1)(2-t)(3-t)(4-t) \psi(t) dt \cdot (5-\alpha)_{r-4} \\ &> 0,\end{aligned}$$

and apply Proposition 3.3 to obtain

$$\begin{aligned} r!J &= H_r \left(\int_1^{\alpha_{r,2}} + \int_{\alpha_{r,2}}^2 \right) (t - \alpha_{r,1})(\alpha_{r,2} - t) \cdots (\alpha_{r,r} - t) dt \\ &> H_r \int_1^2 (t - \alpha_{r,1})(\alpha_{r,1} + 1 - t)(\alpha_{r,3} - t)(\alpha_{r,4} - t) dt \cdot \prod_{k=5}^r (\alpha_{r,k} - \alpha_{r,2}) \\ &> 0. \end{aligned}$$

Thus, we have $\log \Gamma_r(1) < 0$ for $r \geq 4$. Finally, we consider the remaining cases $r = 1, 2, 3$. In these cases, the result follows from $\log \Gamma_r(1) = \log \rho_{r-1} - \log \rho_r$ ($r \geq 1$) and

$$\begin{aligned} \log \rho_0 &= 0, & \log \rho_1 &= \frac{1}{2} \log(2\pi) = 0.91893 \dots, \\ \log \rho_2 &= \frac{1}{2} \log(2\pi) - \zeta'(-1) = 1.0843 \dots, \\ \log \rho_3 &= \frac{1}{2} \log(2\pi) + \frac{\zeta(3)}{8\pi^2} - \frac{3}{2} \zeta'(-1) = 1.1822 \dots \end{aligned}$$

(cf. [11, Theorem 1]).

(v) Put

$$K := (-1)^{r+j-1} r! \int_{\alpha_{r,j}}^{\alpha_{r,j}+1} \binom{t-1}{r} \psi(t) dt, \quad L := (-1)^{j+1} r! \int_{\alpha_{r,j}}^{\alpha_{r,j}+1} F_r(t) dt.$$

Then, we obtain $(-1)^j r! \log \Gamma_r(\alpha_{r,j}) = K + L$ and

$$\begin{aligned} K &= - \int_{\alpha_{r,j}}^{\alpha_{r,j}+1} (t-j)_j (j+1-t)_{r-j} \psi(t) dt \\ &> - \int_j^{\alpha_{r,j}+1} (t-j)_j (j+1-t)_{r-j} (H_j - \gamma) dt \end{aligned}$$

where γ is the Euler constant. Moreover, applying Proposition 3.3, we have

$$L = H_r \int_{\alpha_{r,j}}^{\alpha_{r,j}+1} \prod_{a=1}^{j-1} (t - \alpha_{r,j-a}) \cdot (t - \alpha_{r,j}) \cdot \prod_{b=1}^{r-j} (\alpha_{r,j+b} - t) dt$$

$$\begin{aligned}
&> H_r \int_{\alpha_{r,j}}^{\alpha_{r,j}+1} \prod_{a=1}^{j-1} (t - \alpha_{r,j} + a) \cdot (t - \alpha_{r,j}) \cdot \prod_{b=1}^{r-j} (\alpha_{r,j} + b - t) dt \\
&> \int_j^{\alpha_{r,j}+1} (t-j)_j (j+1-t)_{r-j} H_r dt.
\end{aligned}$$

Hence

$$\begin{aligned}
K + L &> \int_j^{\alpha_{r,j}+1} (t-j)_j (j+1-t)_{r-j} (H_r - H_j + \gamma) dt \\
&> 0.
\end{aligned}$$

Thus, we deduce $\text{sign}(\log \Gamma_r(\alpha_{r,j})) = (-1)^j$ for $j = 2, \dots, r$. \square

5. Extremal values of multiple sine function

In this section, we consider the extremal values of the multiple sine functions.

Proof of Theorem 2. Let r and k be integers with $r \geq 1$ and $0 \leq k \leq r-1$. Since

$$\log S_r(x) = \log S_{r+1}(x) - \log S_{r+1}(x+1)$$

(see [7, Theorem 2.1(a)]), the formula (2.5) shows

$$\begin{aligned}
\log S_r\left(k + \frac{1}{2}\right) &= (-1)^{r-1} \pi \int_{k+1/2}^{k+3/2} \binom{t-1}{r} \cot(\pi t) dt \\
&= (-1)^{r-1} \pi \int_0^{1/2} C_r(k, t) \cot(\pi t) dt
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
C_r(k, t) &= \binom{t+k}{r} - \binom{-t+k}{r} \\
&= (-1)^{r-k-1} (r!)^{-1} \{(t)_{k+1} (1-t)_{r-k-1} + (t)_{r-k} (1-t)_k\}.
\end{aligned}$$

Thus we obtain $\text{sign}(\log S_r(k + 1/2)) = (-1)^k$. \square

Proof of Theorem 3. By Theorem 2 and $S_r(r-x) = S_r(x)^{(-1)^{r-1}}$, it is enough for our purpose to prove

$$\operatorname{sign}\left\{\log S_r\left(k+\frac{1}{2}\right)+\log S_r\left(k+\frac{3}{2}\right)\right\}=(-1)^k \quad (5.2)$$

when $r \geq 3$ and $0 \leq k \leq [(r-3)/2]$. The formula (5.1) shows that

$$\log S_r\left(k+\frac{1}{2}\right)+\log S_r\left(k+\frac{3}{2}\right)=(-1)^{r-1} \pi \int_0^{1/2} D_r(k, t) \cot(\pi t) dt$$

where $D_r(k, t) = C_r(k, t) + C_r(k+1, t)$. We note that

$$\begin{aligned} (-1)^{r-k-1} r! D_r(k, t) &= (r-2-2k-2t)(t)_{k+1}(1-t)_{r-k-2} \\ &\quad + (r-2-2k+2t)(t)_{r-k-1}(1-t)_k. \end{aligned}$$

Hence, $\operatorname{sign} D_r(k, t) = (-1)^{r-k-1}$ for $t \in (0, 1/2)$. Thus, we obtain (5.2) and deduce Theorem 3. \square

Remark. In the similar way, we can get the following results. Let r, k, l be integers with $r \geq 2$ and $1 \leq k, l \leq r-1$. Then

$$\operatorname{sign}(\log S_r(k)) = \begin{cases} (-1)^{k-1} & \text{if } k < r/2, \\ 0 & \text{if } k = r/2, \\ (-1)^k & \text{if } k > r/2. \end{cases}$$

Moreover, $|\log S_r(k)| > |\log S_r(l)|$ when $|k - r/2| > |l - r/2|$.

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